Diekert/Weiß — Sommersemester 2019

Konkrete Mathematik
Aufgabenblatt 2

Besprechung: Dienstag, 18. Juni 2019

1. *Inverse in Monoiden*

Zeigen Sie:

a) In einem **endlichen** Monoid M gilt $st = 1 \implies ts = 1$ für alle $s, t \in M$.

b) Für unendliche Monoide ist diese Aussage im Allgemeinen falsch.

2. Let S be a finite semigroup. Recall that an element $e \in S$ is *idempotent* if it satisfies $e^2 = e$. Show:

a) For each $x \in S$, the set $\{x^k \mid k \geq 1\} \subseteq S$ contains a unique idempotent element. This element is some power x^ℓ with $1 \leq \ell \leq |S|$.

b) The element $x^{|S|!}$ is idempotent for all $x \in S$.

3. *Green's Relations*

Let us define natural partial orders $\leq_\mathcal{L}$, $\leq_\mathcal{R}$, $\leq_\mathcal{J}$ and induced equivalence relations $\sim_\mathcal{L}$, $\sim_\mathcal{R}$, $\sim_\mathcal{J}$. They are given by:

$$\begin{array}{lcclcc}
x \leq_\mathcal{L} y & \iff & Mx \subseteq My & x \sim_\mathcal{L} y & \iff & Mx = My \\
x \leq_\mathcal{R} y & \iff & xM \subseteq yM & x \sim_\mathcal{R} y & \iff & xM = yM \\
x \leq_\mathcal{J} y & \iff & MxM \subseteq MyM & x \sim_\mathcal{J} y & \iff & MxM = MyM
\end{array}$$

Instead of $\sim_\mathcal{L}$, $\sim_\mathcal{R}$, $\sim_\mathcal{J}$ we also write $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{J}$ for the corresponding subsets in $M \times M$. These are three out of five relations referred to as *Green's relations*. There are two more relations $\mathcal{H}$ and $\mathcal{D}$, defined by $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$.

Show that

a) $\mathcal{D} \subseteq \mathcal{J}$ and $\mathcal{R} \circ \mathcal{L} \subseteq \mathcal{J}$.

Show that if M is a finite monoid and s, t be $\mathcal{J}$-equivalent elements (i.e. $s \sim_\mathcal{J} t$), then the following assertions hold:

b) If $s \leq_\mathcal{R} t$ (resp. $s \leq_\mathcal{L} t$), then $s \sim_\mathcal{R} t$ (resp. $s \sim_\mathcal{L} t$).

c) $\mathcal{L}(s) \cap \mathcal{R}(t) \neq \emptyset$, where $\mathcal{L}(s) = \{r \in M \mid r \sim_\mathcal{L} s\}$

d) $\mathcal{J} = \mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$; therefore, $\mathcal{D}$ is an equivalence relation.

Hinweis: Benutzen Sie Aufgabe 2b): es gibt eine Zahl ω, sodass u^ω idempotent ist für alle $u \in M$.

4. *Green's Lemma*

Let M be a finite monoid and $s \in M$. The *local divisor* M_s is defined as follows: It is the set $sM \cap Ms$ together with the multiplication $xs \circ sy = xsy$. The neutral element is s. Note that the $\mathcal{H}$-class $\mathcal{H}(s)$ is a subset of $sM \cap Ms$.

a) Show that $\mathcal{H}(s)$ is exactly the set of units in M_s.

If we endow the set $\mathcal{H}(s)$ with the operation $\circ$, then it becomes a group. It is called the *Schützenberger group* of the $\mathcal{H}$-class $\mathcal{H}(s)$.

Let M be a monoid and s, t be $\mathcal{R}$-equivalent elements such that $s = tu$ and $t = sv$. Then the right multiplication $\cdot v$, mapping x to xv, induces a mapping $Ms \to Mt$, $x \mapsto xv$. Show that the mapping $\cdot v$ enjoys the following properties.

b) It induces an isomorphism between the local divisors M_s and M_t.

c) It maps $\mathcal{L}(s)$ bijectively onto $\mathcal{L}(t)$.

d) If M is finite, then $\mathcal{L}(s) \xrightarrow{\cdot v} \mathcal{L}(t)$ respects $\mathcal{H}$-classes. More precisely, if M is finite, then $\mathcal{H}(x) \cdot v = \mathcal{H}(xv)$ for $x \in \mathcal{L}(s)$.

Now, let M be finite and s, t $\mathcal{J}$-equivalent. Conclude the following:

e) The local divisors M_s and M_t are isomorphic.

f) The Schützenberger groups $\mathcal{H}(s)$ and $\mathcal{H}(t)$ are isomorphic.

g) If $e \in M$ is an idempotent, then $\mathcal{H}(e)$ is a subgroup in the monoid M.